Name:	
Class:	

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION 2015

MATHEMATICS EXTENSION 1

General Instructions:

- Reading Time: 5 minutes.
- Working Time: 2 hours.
- Write in black or blue pen.
- Board approved calculators & templates may be used
- A Standard Integral Sheet is provided.
- In Question 11 - 14, show all relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or badly arranged working.

Total Marks 70

Section I: 10 marks

- Attempt Question 1 – 10.
- Answer on the Multiple Choice answer sheet provided.
- Allow about 15 minutes for this section.

Section II: 60 Marks

- Attempt Question 11 - 14
- Answer on blank paper unless otherwise instructed. Start a new page for each new question.
- Allow about 1 hours & 45 minutes for this section.

The answers to all questions are to be returned in separate *stapled* bundles clearly labelled Question 11, Question 12, etc. Each question must show your Candidate Number.

Section 1 (10 marks)

Attempt questions 1 -10. Use the multiple-choice answer sheet provided.

1. Evaluate $\lim_{x \to 0} \frac{3\sin 7x}{5x}$

(A) 3 (B) 0 (C) $\frac{21}{5}$ (D) $\frac{15}{7}$

2. For what values of x is $\frac{x+4}{x-1} < 6$?

(A) 

(B) 

(C) 

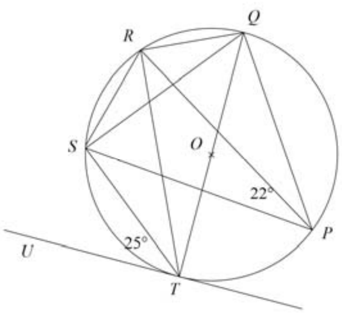
(D) 

3. The interval joining the points $A(-3,2)$ and $B(-9,y)$ is divided externally in the ratio 5:3 by the point $P(x,-13)$. What are the values of x and y?

(A) $x=-27,\ y=22$ (B) $x=-18,\ y=-4$

(C) $x=6,\ y=12$ (D) $x=27,\ y=4$

4. A circle with centre O has a tangent TU, diameter QT, $\angle STU = 25^\circ$ and $\angle RPS = 22^\circ$.



What is the size of $\angle RTQ$?

(A) 22°

(B) 25°

(C) 43°

(D) 47°

5. For the polynomial equation $6-4x+10x^2-8x^3=0$, the sum of its roots, when divided by the product of its roots would be:

(A) $\frac{5}{3}$ (B) $\frac{-4}{3}$ (C) $\frac{-1}{2}$ (D) $\frac{5}{4}$

6. A particle moves such that when it is *x* metres from the origin its acceleration is given by $a=-\frac{1}{2}e^{-x}$. What is its velocity when $x=3$, given that $v=1$ when $x=0$?

(A) $0.050\ \text{ms}^{-1}$ (B) $0.070\ \text{ms}^{-1}$ (C) $0.158\ \text{ms}^{-1}$ (D) $0.223\ \text{ms}^{-1}$

7. Which of the following is the correct expression for $\int \frac{dx}{\sqrt{36-x^2}}$?

(A) $\cos^{-1}\frac{x}{6}+c$

(B) $\cos^{-1}6x+c$

(C) $\sin^{-1}\frac{x}{6}+c$

(D) $\sin^{-1}6x+c$

8. Eden, Toby and four friends arrange themselves at random in a circle. What is the probability that Eden and Toby are *not* together?

(A) $\frac{1}{120}$ (B) $\frac{2}{5}$ (C) $\frac{3}{5}$ (D) $\frac{119}{120}$

9. If $t=\tan\frac{\theta}{2}$ which of the following expressions is equivalent to $4\sin\theta+3\cos\theta+5$?

(A) $\frac{2(t+2)^2}{1-t^2}$ (B) $\frac{(t+4)^2}{1-t^2}$ (C) $\frac{2(t+2)^2}{1+t^2}$ (D) $\frac{(t+4)^2}{1+t^2}$

10. An expression for the general solution to the trigonometric equation $\tan 3x=-\sqrt{3}$ where n is any integer is:

(A) $x=\frac{n\pi}{3}-\frac{\pi}{9}$ (B) $x=\frac{n\pi}{3}-\frac{\pi}{3}$

(C) $x=\frac{n\pi}{3}+\frac{\pi}{3}$ (D) $x=\frac{n\pi}{3}-\frac{2\pi}{9}$

Section II (60 marks)

Attempt all questions from 11-14. Answer each question on a separate page.

Question 11 (15 marks)

(a) The number of animals in a local farm who will be infested with a virus adheres to the equation

$n = \dfrac{p}{1 + Ce^{-kt}}$ where n = the number of animals infested by the virus

p = the total number of animals

k = the growth constant

t = the time in months

C = constant

The farmer notices that initially 1 animal out of the animal population of 200 is infested with the virus. After one month the number of animals infested with the virus increases to 5.

(i) Show that after t months, $n = \dfrac{200}{1 + 199e^{-kt}}$ 1

(ii) Show that k = 1.63 (to 3 significant figures) 1

(iii) How many animals can the farmer expect to be infested after 3 months. 2

(b) (i) Find $\dfrac{d}{dx}\left\{\dfrac{2x}{4 + x^2} + \tan^{-1}\left(\dfrac{x}{2}\right)\right\}$ 2

(ii) Hence evaluate $\displaystyle\int_0^2 \frac{dx}{\left(4 + x^2\right)^2}$ 3

(c) A spherical metal ball is being heated such that the volume increases at a rate of $5\pi\ mm^3/\min$. At what rate is the surface area increasing when the radius is 3 mm. 3

(d) Find an expression for $\displaystyle\int \frac{e^{3x}}{1 + e^x}dx$ using the substitution $u = 1 + e^x$. 3

Question 12 (15 marks) Start a new page

(a) A group of 15 students from a local school is selected for training in soccer to represent the school at grade sport. However only a team of 11 players is to be chosen for the Wednesday game.
The probability that a player will not be available to play on Wednesday due to injury or other commitments is 0.14.

(i) Find the probability that 3 students will not be available for the Wednesday grade sport in soccer. Answer to 3 decimal places. 2

(ii) Write the numerical expression for the probability that the team will be unable to make up a team of all fit 11 players. You do not have to simplify the answer. 2

(b) Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be two points on the parabola $x^2 = 4ay$. The secant PQ passes through the point $A(a,0)$, and the tangents at P and Q meet at R.

(i) Show that $p+q=2pq$. 2

(ii) Find the coordinates of R in terms of p and q. 3

(iii) As P and Q vary, show that R moves on a straight line. 1

(iv) Find the restrictions on the x values of the locus of R. 1

(c) Use mathematical induction to prove that for all integers $n \geq 3$, 4

$$\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\ldots\ldots\ldots\left(1-\frac{2}{n}\right)=\frac{2}{n(n-1)}.$$

Question 13 (15 marks) Start a new page

(a) (i) Using the auxiliary angle method express $3\sin 2t + 2\cos 2t$ in the form $r\sin(2t+\alpha)$. 2

A particle moves horizontally in a straight line so that its position x from a fixed point at time t is given by:

$$x = 3\sin 2t + 2\cos 2t - 2$$

Displacement is measured in metres and time in hours.

(ii) Find an equation to represent the acceleration of this particle and prove that it is moving in simple harmonic motion. 2

(iii) Given that the particle is at the origin at noon, between what times will the particle be more than one metre to the right of the origin for the first time (Let the time at $t=0$ be noon). Give your times correct to the nearest minute. 2

(b) Consider the function $y = \frac{1}{2}\cos^{-1}(x-1)$.

(i) Find the domain and range of the function. 2

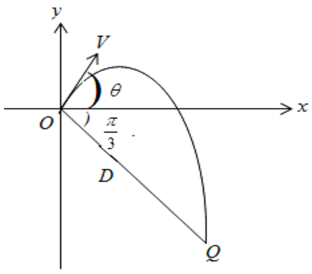
(ii) Sketch the graph of the function showing clearly the coordinates of the end points. 1

(iii) The region in the first quadrant bounded by the curve $y = \frac{1}{2}\cos^{-1}(x-1)$ and the coordinate axes is rotated about the y axis. Find the volume of the solid of revolution, giving your answer in simplest exact form. 3

(c) What is the exact value of the definite integral $\int_0^{\pi} 3\sin^2 \frac{x}{4}\, dx$? 3

Question 14 (15 marks) **Start a new page**

(a) In a BMX dirt bike competition the take-off point O for each competitor was located at the top of the downslope. The angle between the downslope and the horizontal is $\frac{\pi}{3}$. The biker takes off from O with velocity V m/s at an angle θ to the horizontal, where $0 \le \theta \le \frac{\pi}{2}$. The biker lands on the downslope at some point Q, a distance D metres from O.



The flight path of the biker is given by

$x = Vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + Vt\sin\theta$

where t is the time in seconds after take-off.
(DO NOT PROVE THIS)

(i) Show that the Cartesian equation of the flight path of the biker is given by 2

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta.$$

(ii) Show that 3

$$D = 4\frac{V^2}{g}\cos\theta\left(\sqrt{3}\cos\theta + \sin\theta\right)$$

(iii) Show that 2

$$\frac{dD}{d\theta} = 4\frac{V^2}{g}\left(\cos 2\theta - \sqrt{3}\sin 2\theta\right)$$

(iv) Show that D has a maximum value and find the value of θ for which this occurs. 3

(b) (i) Considering the identity $(1-x)^n(1+x)^n \equiv \left(1-x^2\right)^n$, where n is a positive integer, show that for integer values of r, 2

$$\sum_{k=0}^{2r}(-1)^k \; {}^nC_k \, {}^nC_{2r-k} = (-1)^r \, {}^nC_r \text{ provided } 0 \le r \le \tfrac{1}{2}n.$$

(ii) Hence show that $\sum_{k=0}^{r}(-1)^k \; {}^nC_k \, {}^nC_{2r-k} = \frac{1}{2}(-1)^r \, {}^nC_r\left\{1 + {}^nC_r\right\}$ for $0 \le r \le \frac{1}{2}n$. 2

(iii) Hence evaluate $\sum_{k=0}^{6}(-1)^k \left({}^{12}C_k\right)^2$ as a basic numeral. 1

END OF THE EXAMINATION

Multiple Choice

1. $\lim_{x\to 0} \frac{3\sin 7x}{5x} = \frac{3}{5} \lim_{x\to 0}\left[\frac{\sin 7x}{7x}\times 7\right] = \frac{3}{5}\times 1\times 7 = \frac{21}{5}$ → C

2. $\frac{x+4}{x-1} < 6 \Rightarrow \frac{x+4}{x-1}(x-1)^2 < 6(x-1)^2$

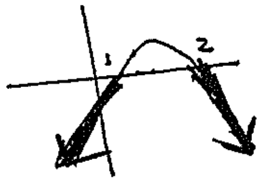
$(x+4)(x-1) - 6(x-1)^2 < 0$

$(x-1)[(x+4) - 6(x-1)] < 0$

$(x-1)[x+4-6x+6] < 0$

$(x-1)(-5x+10) < 0$

$x = 1$ or $x = 2$



$x < 1$ and $x > 2$

D

3. A $(-3, 2)$ B $(-9, y)$ ratio $(-5:3)$ externally and P $(x, -13)$

$x = \frac{-5\times -9 + 3\times -3}{-5+3} = \frac{45-9}{-2} = -18$

$y = -13 = \frac{-5y+6}{-5+3}$

$\therefore \frac{-5y+6}{-2} = -13$

$y = -4$

$\therefore x = -18$ and $y = -4$

B

4. $\angle RPS = \angle STR = 22^\circ$ (angle in same segment standing on same arc are equal)

$\therefore UTO = 90$ (tangent to radius at pt of contact).

$\angle RTQ + \angle STU + \angle STR = 90$

$RTQ + 25 + 22 = 90$

$\therefore RTQ = 43^\circ$

C

5. $6 - 4x + 10x^2 - 8x^3 = 0$

$\frac{\text{Sum of roots}}{\text{product of roots}} = -\frac{b}{a} \div -\frac{d}{a}$

$= \frac{-b}{a}\times\frac{a}{-d}$

$= \frac{b}{d} = \frac{10}{6} = \frac{5}{3}$

A

6. $a = -\frac{1}{2}e^{-x} \Rightarrow v = \int -\frac{1}{2}e^{-x}\,dx$

$v^2 = 2\int -\frac{1}{2}e^{-x}\,dx$

$= e^{-x} + C$

when $x = 0$, $v = 1$ $\therefore C = 0$ (as $1 = e^0 + C$)

$\therefore v^2 = e^{-x}$

$v = \sqrt{e^{-x}}$ $(v > 0)$

when $x = 3$ $v = \sqrt{e^{-3}} = 0.223$ u/s.

D

$$\int \frac{dx}{\sqrt{36-x^2}} = \sin^{-1}\left(\frac{x}{6}\right)+c$$

C

Possible arrangements in circle is $(n-1)! = (6-1)! = 5!$

favourable arrangement $= 5! - 2\times4!$

$= 72$

$\therefore\ P = \frac{72}{120} = \frac{3}{5}$

C

$4\sin\theta + 3\cos\theta + 5$

$4\left(\frac{2t}{1+t^2}\right) + 3\left(\frac{1-t^2}{1+t^2}\right) + 5$

$$\frac{8t + 3 - 3t^2 + 5(1+t^2)}{1+t^2} = \frac{8t+3-3t^2+5+5t^2}{1+t^2} = \frac{2t^2+8t+8}{(1+t)^2}$$

$$= \frac{2(t^2+4t+4)}{1+t^2} = \frac{2(t+2)^2}{1+t^2}$$

C

$\tan x = -\sqrt{3}$ and $\tan\left(-\frac{\pi}{3}\right) = \sqrt{3}$

$3x = n\pi - \frac{\pi}{3}$

$3x = \frac{\pi}{3}(3n-1)$

$x = \frac{\pi}{9}(3n-1)$

$= \frac{n\pi}{3} - \frac{\pi}{9}$

A

MATHEMATICS Extension 1 : Question 11

Suggested Solutions	Marks	Marker's Comments
(a) when $t=0$, $n=1$, $p=200$ $1=\dfrac{200}{1+Ce^0}$ $\therefore 1+C=200$ $C=199$ $\therefore n=\dfrac{200}{1+199e^{-kt}}$	1	well done.
(b) when $t=1$ $n=5$ $5=\dfrac{200}{1+199e^{-k}}$ $5+5(199e^{-k})=200$ $e^{-k}=\dfrac{200-5}{5(199)}$ $=\dfrac{39}{199}$ $\therefore k=-\ln\dfrac{39}{199}$ or $\ln\dfrac{199}{39}$ ≈ 1.63	1.	
$t=3$ $n=\dfrac{200}{1+199\left(e^{39/199}\right)^3}$ $=80.06$ $\therefore$ 80 animals infected	1 1	

MATHEMATICS Extension 1 : Question……..

Suggested Solutions	Marks	Marker's Comments
(b) (i) $\frac{d}{dx}\left[\frac{2x}{4+x^2}+\tan^{-1}\frac{x}{2}\right]$ $= \frac{2(4+x^2)-2x\times 2x}{(4+x^2)^2}+\frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2}$ $= \frac{8-2x^2+2(4+x^2)}{(4+x^2)^2}$ $= \frac{16}{(4+x^2)^2}$	2	1mark for each differentiation
(ii) $\int_0^2 \frac{dx}{(4+x^2)^2} = \frac{1}{16}\int_0^2 \frac{16\,dx}{(4+x^2)^2}$ $= \frac{1}{16}\left[\frac{2x}{4+x^2}+\tan^{-1}\frac{x}{2}\right]_0^2$ $= \frac{1}{16}\left[\frac{2\times 2}{4+4}+\tan^{-1}\frac{2}{2}-\{0+\tan^{-1}0\}\right]$ $= \frac{1}{16}\left[\frac{\pi}{4}+\frac{1}{2}\right]$	1 1 1	Students who got incorrect answer in (i) had difficulty with this part. - poor ~~and~~ substitution careless errors.

MATHEMATICS Extension 1 : Question……11

Suggested Solutions	Marks	Marker's Comments
c) $SA = 4\pi r^2$ $\quad V = \frac{4}{3}\pi r^3$ $\frac{dA}{dr} = 8\pi r$ $\quad \frac{dV}{dr} = 4\pi r^2$	1	① Some students didn't know correct formulae.
$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ $= 8\pi r \times \frac{dr}{dt}$ $\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$ $= 5\pi \times \frac{1}{4\pi r^2}$ $= \frac{5}{4r^2}$	1	② students tried to use $V = \frac{1}{3}Ar$ but A and r were both variables. ③ some found relationship for r and made question much harder.
$\therefore \frac{dA}{dt} = 8\pi r \times \frac{5}{4r^2}$ $= \frac{10\pi}{r}$ when $r = 3$ $= \frac{10\pi}{3}$ mm²/min	1	
$\int \frac{e^{3x}}{1+e^x}\,dx$ $\quad$ let $u = 1+e^x \rightarrow e^x = u-1$ $du = e^x\,dx$ $= \int \frac{e^{2x}e^x\,dx}{1+e^x}$ $= \int \frac{(u-1)^2}{u}\,du$	1	Needed to get to u integral without mixing u and x for first mark.

MATHEMATICS Extension 1 : Question........

Suggested Solutions	Marks	Marker's Comments
$= \int \frac{u^2 - 2u + 1}{u}\, du$		
$= \int u - 2 + \frac{1}{u}\, du.$		
$= \frac{u^2}{2} - 2u + \ln u + c$	1	* A number of students failed to substitute back
$= \frac{(1+e^x)^2}{2} - 2(1+e^x) + \ln(1+e^x) + c$	1	* NB any further expansion was not required

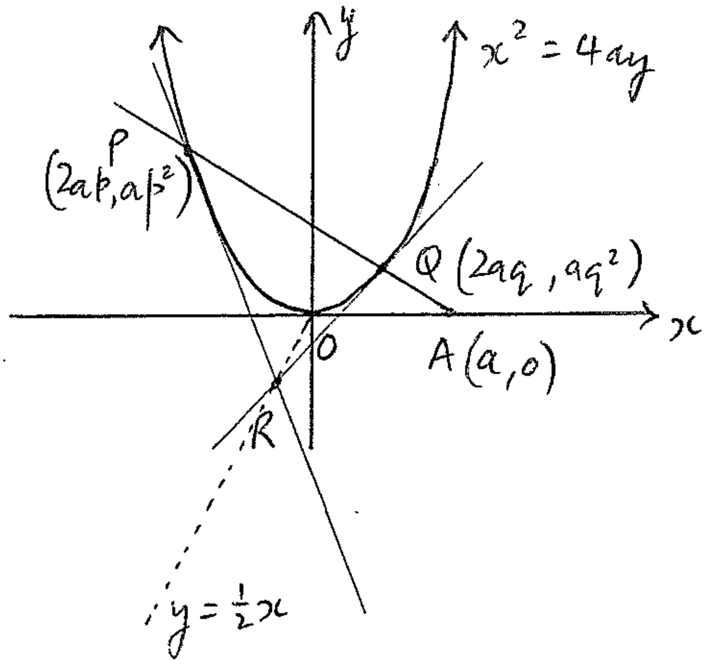
3U 2015 TRIAL MATHEMATICS Extension 1 : Question 12 JRAHS

Suggested Solutions	Marks	Marker's Comments
QUESTION 12		
a) (i) $P = {}^nC_r\, p^r q^{n-r}$		
$= {}^{15}C_3 (0.14)^3 (0.86)^{12}$	1	for recognising binomial probability
$= 0.20435\ldots$		
$= 0.204$ (3dp)	1	for correct final answer
		(If ${}^{11}C_3$, max 1)
(ii) $P(\text{no team}) = P(5 \text{ not playing}) + P(6 \text{ not playing}) + \cdots + P(15 \text{ not playing})$	1	correct verbal expression
$= \sum_{i=5}^{15} {}^{15}C_i (0.14)^i (0.86)^{15-i}$	1	correct numerical expression
OR $P(\text{no team}) = 1 - P(\text{available})$		
$= 1 - P(15 \text{ play}) + \cdots + P(11 \text{ play})$		
$= 1 - \sum_{i=0}^{4} {}^{15}C_i (0.14)^i (0.86)^{15-i}$		
$= 1 - 0.952$		not required!
$= 0.0478$ (4dp)		

OR

$P(\text{no team}) = 1 - \sum_{i=11}^{15} {}^{15}C_i (0.14)^{15-i} (0.86)^i$

3u 2015 TRIAL MATHEMATICS Extension 1 : Question 12 JRAHS

Suggested Solutions	Marks	Marker's Comments
b) 		
(i) $m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq}$ $= \dfrac{a(p-q)(p+q)}{2a(p-q)}$ $= \frac{1}{2}(p+q)$	1	for correct gradient of line PQ
Equation of PQ: $y - ap^2 = \frac{1}{2}(p+q)(x-2ap)$ $= \frac{1}{2}(p+q)x - apq - ap^2$ Sub $A(a,0)$: $0 - ap^2 = \frac{1}{2}(p+q)a - apq - ap^2$ $= ap + aq - 2apq$ $\therefore p+q = 2pq$, $a \neq 0$	1	for obtaining equation for PQ and substituting $A(a,0)$
	OR	$m_{PQ} = m_{QA}$

3U 2015 TRIAL **MATHEMATICS Extension 1 : Question** 12 JRAHS

Suggested Solutions	Marks	Marker's Comments
(ii) Gradient of PR: $y = \frac{1}{4a}x^2$ $y' = \frac{x}{2a}$ when $x = 2ap$, $y' = p$ $\therefore m_{PR} = p$ Equation of PR: $y - ap^2 = p(x - 2ap)$ or $y = px - ap^2 \cdots (1)$		
Similarly Equation of QR: $y = qx - aq^2 \cdots (2)$	1	for correct equations for PR and QR
(1)=(2): $px - ap^2 = qx - aq^2$ $x(p-q) = a(p^2 - q^2)$ $\therefore x = a(p+q) \cdots (3)$	1	correct x-coordinate
Sub (3) into (1): $\therefore y = pa(p+q) - ap^2$ $= apq \cdots (4)$	1	correct y-coordinate
$\therefore R(a(p+q), apq)$ OR $R(2apq, 2ap^2q - ap^2)$ using $p+q = 2pq$		

3U 2015 TRIAL **MATHEMATICS Extension 1 : Question** 12 JRAHS

Suggested Solutions	Marks	Marker's Comments
(iii) y-coordinate of R: $y = apq$ But $p+q = 2pq$ ... (proved above) $\therefore y = \frac{1}{2}a(p+q)$ But $x = a(p+q)$ $\therefore y = \frac{1}{2}x$ Hence R moves on a straight line	1	For correct equation
(iv) Method I: Sub $x = 2y$ into $x^2 = 4ay$ i.e. $4y^2 = 4ay$ $y(y-a) = 0$ $\therefore y = 0$ or a $\Rightarrow x = 0$ or $2a$ Since R has to be outside of the parabola (for the tangents to meet), $x < 0$ or $x > 2a$	1	For correct inequalities

Suggested Solutions	Marks	Marker's Comments
(iv) Method II:		

From $p+q=2pq$, $q=\frac{p}{2p-1}$

Sub into $x=a(p+q)$

$$=\frac{2ap^2}{2p-1}$$

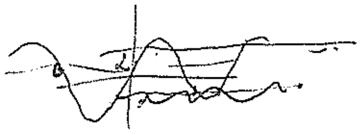
$\Rightarrow \quad 2a(p^2)-2x(p)+x=0$

$\hookrightarrow$ quadratic in p.

$\Delta > 0$

$\Rightarrow \quad x^2-2ax>0$

$\Rightarrow \quad \underline{x<0 \text{ or } x>2a}$



Question 13

a) (i). $3\sin 2t + 2\cos 2t$

$r^2 = 2^2 + 3^2$

$r = \sqrt{13}$ 1

and

$\tan^{-1}\alpha = \frac{b}{a} = \frac{2}{3}$

$\therefore \alpha = \tan^{-1}\left(\frac{2}{3}\right)$ 1

$= \alpha = 0.588202603$

$\therefore \alpha = 0.59$ to 2dp

or $\alpha = 33°41'$

$\therefore 3\sin 2t + 2\cos 2t = \sqrt{13}\sin(2t + 0.59)$

(ii) $x = 3\sin 2t + 2\cos 2t - 2.$

$\dot{x} = 6\cos 2t - 4\sin 2t$

$\ddot{x} = -12\sin 2t - 8\cos 2t$

$= -4(3\sin 2t + 2\cos 2t)$ 1

$= -4(3\sin 2t + 2\cos 2t - 2 + 2)$

$= -4(x+2)$

This is in the form $-n^2(x+c)$ } 1

$\therefore$ SHM.

$2 \cdot 9.88$

$x = \sqrt{13}\sin(2t + \tan^{-1} 2/3) - 2$

$\dot{x} = 2\sqrt{13}\cos(2t + \tan^{-1} 2/3)$

1 $\ddot{x} = 4\sqrt{13}\sin(2t + \tan^{-1} 2/3)$

$\ddot{x} = 4\sqrt{13}\sin(2t + \tan^{-1} 2/3) - 8 +$

$= 4[\sqrt{13}\sin(2t + \tan^{-1} 2/3) - 2 + 2]$

$= 4[x+2]$

(iii). Initially at $t=0$ (at noon) the particle is at the origin

Times when particle is greater than 1 is:

$3\sin 2t + 2\cos 2t - 2 = 1$

$3\sin 2t + 2\cos 2t = 3.$

From part (i)

$3\sin 2t + 2\cos 2t = \sqrt{13}\sin(2t + 0.59)$

$\therefore \sqrt{13}\sin(2t + 0.59) = 3$

$\sin(2t + 0.59) = \frac{3}{\sqrt{13}}$

$\therefore 2t + 0.59 = \sin^{-1}\left(\frac{3}{\sqrt{13}}\right)$ 1

$2t + 0.59 = 0.982793723$ or $\pi - 0.982793723 = 2.15879893$

$\therefore 2t = 0.394791197$ or $= 1.570796327$

$t = 0.197395559$ or $= 0.785398163$

$\therefore t \approx 12$ min or $t = 47$ min

$\therefore$ particle will be greater than 1m between 1

12.12 pm and 12.47 pm.

$\sin^{-1}\left(\frac{1}{\sqrt{13}}\right) = 2t + \tan^{-1} 2/3$ 2.59r $\longrightarrow$ maximum 1 mark

13 b (i) $y = \frac{1}{2}\cos^{-1}(x-1)$

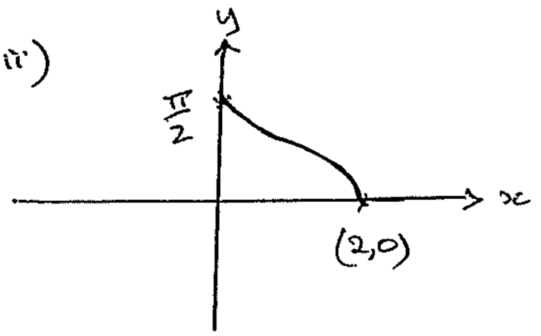
$\therefore -1 \le (x-1) \le 1$

Domain is $0 \le x \le 2$ (1)

Range is $0^\circ \le \cos^{-1}(x-1) \le \pi$ (1)

$0^\circ \le y \le \frac{\pi}{2}$.

b (ii)



must show endpoints and have correct shape for (1)

(iii). $x = 1 + \cos 2y$

$\therefore x^2 = (1+\cos 2y)^2 = 1 + 2\cos 2y + \cos^2 2y$

$= 1 + 2\cos 2y + \frac{1}{2}(1 + \cos 4y)$ (1)

$V = \pi \int_0^{\pi/2} (1 + 2\cos 2y + \frac{1}{2}(1 + \cos 4y)\, dy$

$= \pi \int_0^{\pi/2} ((\frac{3}{2} + 2\cos 2y + \frac{1}{2}\cos 4y))\, dy$

$= \pi \left[\frac{3}{2}y + \sin 2y + \frac{1}{8}\sin 4y\right]_0^{\pi/2}$. (1)

$= \pi \left[\frac{3\pi}{4} + \sin \pi + \frac{1}{8}\sin 2\pi\right] - \left[0 - \sin 0 + \frac{1}{8}\sin 0\right]$ (2)

$= \pi \left[\left(\frac{3\pi}{4} + 0 + 0\right) - 0\right)$ 3rd mark.

$= \frac{3\pi^2}{4}$

$\therefore$ Volume is $\frac{3\pi^2}{4}$ unit^3 (1)

$\int_0^{\pi/4} \ldots$

$= \left(\pi + \frac{3\pi}{8}\right)$

max (2) marks!!

(c) $\displaystyle\int_0^{\pi} 3\sin^2 \frac{x}{4}\,dx.$

we know $\displaystyle\int \sin^2 x = \frac{1}{2}(1-\cos 2x)\,dx.$

$$= \int_0^{\pi} 3\cdot\frac{1}{2}\left(1-\cos\frac{2x}{4}\right)dx \quad 1$$

$$= \frac{3}{2}\int_0 \left(1-\cos\frac{x}{2}\right)dx$$

$$= \int_0^{\pi} \frac{3}{2}\left(1-\cos\frac{\cancel{2}x}{2}\right)dx$$

$$= \frac{3}{2}\left[x - \frac{\sin x/2}{1/2}\right]_0^{\pi}.$$

$$= \frac{3}{2}\left[x - 2\sin\frac{x}{2}\right]_0^{\pi}. \quad 1$$

$$= \frac{3}{2}\left[\left(\pi - 2\sin\frac{\pi}{2}\right) - 0\right]$$

$$= \frac{3}{2}\left[(\pi-2) - 0\right]$$

$$= \frac{3\pi}{2} - 3 \quad 1$$

MATHEMATICS Extension 1 : Question 14

Suggested Solutions	Marks	Marker's Comments

Question 14

a) (i) $x = vt\cos\theta$

$t = \dfrac{x}{v\cos\theta} \longrightarrow$ eq (1)

$y = -\frac{1}{2}gt^2 + vt\sin\theta \longrightarrow$ eq (2)

Sub (1) in (2)

$$y = -\frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2 + v\left(\frac{x}{v\cos\theta}\right)\sin\theta$$

$$= \frac{-gx^2}{2v^2\cos^2\theta} + v\left(\frac{x}{v\cos\theta}\right)\sin\theta$$

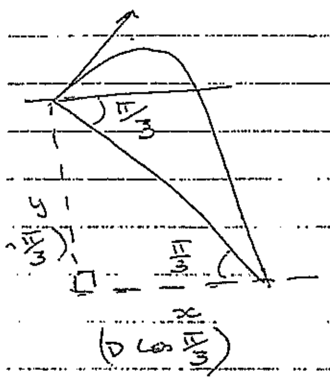
$$= \frac{-gx^2}{2v^2\cos^2\theta} + x.\tan\theta$$

$$= x\tan\theta - \frac{gx^2}{2v^2}\sec^2\theta$$

(1) $t = \dfrac{x}{v\cos\theta}$

(1) complete proof

(ii)



$x = D\cos\theta$

$x = D\cos\frac{\pi}{3}$

$= D \times \frac{1}{2}$

$\therefore x = \frac{D}{2}$

$y = -D\sin\theta$

$= -D\frac{\sqrt{3}}{2}$

$\therefore y = -D\frac{\sqrt{3}}{2}$ (1)

or $y = -\sqrt{3}x$

Sub in eq in part 1

$$y = x\tan\theta - \frac{gx^2}{2v^2}.\sec^2\theta.$$

$$\frac{-D\sqrt{3}}{2} = \frac{D}{2}\tan\theta - \frac{g\left(\frac{D}{2}\right)^2}{2v^2}.\sec^2\theta.$$

$$\frac{-\sqrt{3}D}{2} = \frac{D}{2}\tan\theta - \frac{gD^2}{8v^2}.\sec^2\theta.$$

MATHEMATICS Extension 1 : Question 14

Suggested Solutions	Marks	Marker's Comments
$\frac{gD^2}{8V^2}\cdot\sec^2\theta = \frac{D}{2}\tan\theta + \frac{\sqrt{3}}{2}D$		
$\frac{gD^2}{8V^2}\cdot\sec^2\theta = \frac{1}{2}D(\tan\theta + \sqrt{3})$		
$\frac{gD}{8V^2}\cdot\sec^2\theta = \frac{1}{2}(\tan\theta + \sqrt{3})$		
$\therefore D = \dfrac{\frac{1}{2}8V^2(\tan\theta + \sqrt{3})}{g\sec^2\theta}$		(1) D or x in terms of $\tan\theta$.
$D = \frac{4V^2}{g}\cdot\cos^2\theta(\tan\theta + \sqrt{3})$		
$D = \frac{4V^2}{g}(\cos^2\theta\tan\theta + \sqrt{3}\cos^2\theta)$		
$D = \frac{4V^2}{g}(\sin\theta\cos\theta + \sqrt{3}\cos^2\theta)$		
$D = \frac{4v^2}{g}\cos\theta(\sin\theta + \sqrt{3}\cos\theta)$.		(1) correct solution
(iii) $D = \frac{4V^2}{g}[\cos\theta(\sin\theta + \sqrt{3}\cos\theta)]$		
$= \frac{2v^2}{g}(2\sin\theta\cos\theta + 2\sqrt{3}\cos^2\theta)$	(2)	(1) Derivative
$= \frac{2v^2}{g}(\sin 2\theta + \sqrt{3}(2\cos^2\theta))$		(1) change to 2θ terms.
$= \frac{2v^2}{g}(\sin 2\theta + \sqrt{3}(1 + \cos 2\theta))$		
$\frac{dD}{d\theta} = \frac{2v^2}{g}(2\cos 2\theta - 2\sqrt{3}\sin 2\theta)$		
$= \frac{4v^2}{g}(\cos 2\theta - \sqrt{3}\sin 2\theta)$		

MATHEMATICS Extension 1 : Question 14

Suggested Solutions	Marks	Marker's Comments

(b)

(i) $(1-x)^n = 1 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 \ldots (-1)^k\, {}^nC_k x^k \ldots (-1)^n\, {}^nC_n x^n$

$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 \ldots {}^nC_k x^k \ldots {}^nC_n x^n$

(1) 3 expansions

$(1-x)^n(1+x)^n$

Terms in $x^{2r} = {}^nC_0\, {}^nC_{2r} - {}^nC_1\, {}^nC_{2r-1} + {}^nC_2\, {}^nC_{2r-2} - {}^nC_3\, {}^nC_{2r-3} + \ldots {}^nC_{2r}\, {}^nC_0$

$$= \sum_{k=0}^{2r} {}^nC_k\, {}^nC_{2r-k} (-1)^k$$

Note:

$0 \le 2r \le n$, $0 \le r \le \frac{n}{2}$ for inclusion in series

provided $2r < n$, k can take all the values $0, 1, 2 \ldots 2r$ when forming the product

RHS $(1-x^2)^n = 1 - {}^nC_1 x^2 + {}^nC_2 x^4 - {}^nC_3 x^6 \ldots (-1)^k\, {}^nC_k x^{2k} \ldots$

provided $0 \le r \le \frac{1}{2}n$

$\therefore$ coefficient of x^{2r} is $(-1)^r\, {}^nC_r$

(1) taking x^{2r} term and showing series of ${}^nC_r\, {}^nC_{2r-k}$ terms and complete proof

$$\therefore \sum_{k=0}^{2r} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} = (-1)^n\, {}^nC_r$$

b(ii)

$$\sum_{k=0}^{2r} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} = (-1)^r\, {}^nC_r \quad 0 \le r \le \tfrac{1}{2}n \text{ from (i)}.$$

$$\therefore \sum_{k=0}^{r-1} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} + (-1)^r\, {}^nC_r\, {}^nC_r + \sum_{k=r+1}^{2r} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} = (-1)^r\, {}^nC_r$$

$$\sum_{k=0}^{r-1} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} + (-1)^r\, {}^nC_r\, {}^nC_r + \sum_{k=0}^{r-1} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} = (-1)^r\, {}^nC_r.$$

$$2\sum_{k=0}^{r-1} (-1)^k\, {}^nC_k\, {}^nC_{2r-k} + 2(-1)^r\, {}^nC_r\, {}^nC_r = (-1)^r\, {}^nC_r + (-1)^r\, {}^nC_r\, {}^nC_r.$$

$$2\sum_{k=0}^{r} (-1)\, {}^nC_k\, {}^nC_{2r-k} = (-1)^r\, {}^nC_r\left[1 + (-1)^r\, {}^nC_r\right]$$

$$\sum_{k=0}^{r} (-1)\, {}^nC_k\, {}^nC_{2r-k} = \tfrac{1}{2}(-1)^r\, {}^nC_r\left[1 + (-1)^r\, {}^nC_r\right]$$

(1) Recognizing there was only one middle term ${}^nC_r\, {}^nC_r (-1)^r$ and other 2 parts of series were the same.

(1) complete proof.

MATHEMATICS Extension 1 : Question 14

Suggested Solutions	Marks	Marker's Comments
(iv) Max when $\frac{dD}{d\theta} = 0$		
ie $\cos 2\theta - \sqrt{3}\sin 2\theta = 0$		
$\cos 2\theta = \sqrt{3}\sin 2\theta$		
$\frac{\cos 2\theta}{\sin 2\theta} = \sqrt{3}$		
$\tan 2\theta = \frac{1}{\sqrt{3}}$		
$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$		
$\therefore 2\theta = \pi/6$		
$\theta = \pi/12$		
$\therefore$ Max occurs at $\theta = \frac{\pi}{12} \approx 15^\circ$	1	$\theta = \frac{\pi}{12}$.
Test for max.		
derivative table.		(2) Table. must include or refer to $\frac{v^2}{g}$.

θ	$\pi/13$	$\pi/12$	$\frac{\pi}{11}$
$\frac{dD}{d\theta}$	$+0.08 \times 4v^2/g$	0	$-0.095 \times 4v^2/g$

Suggested Solutions	Marks	Marker's Comments
local max at $\theta = \pi/12$.		
one stationary point for $0 < \theta < \pi/2$ $\therefore$ $\theta = \pi/12$ gives max D.		OR
OR second derivative		
$\frac{d^2D}{d\theta^2} = \frac{4v^2}{g}\left[-2\sin 2\theta - 2\sqrt{3}\cos 2\theta\right]$		(1) 2nd derivative
When $\theta = \pi/12$.		
$\frac{d^2D}{d\theta^2} = \frac{4v^2}{g}\left[\frac{-2}{2} - 2\sqrt{3} \times \frac{\sqrt{3}}{2}\right]$		(1) value of $\frac{d^2D}{d\theta^2}$.
$= \frac{-16v^2}{g} < 0$		
$\therefore$ concave down $\therefore$ local max.		
one stat point $0 < \theta < \pi/2$ $\therefore$ abs max		

MATHEMATICS Extension 1 : Question 14

Suggested Solutions	Marks	Marker's Comments
$\frac{gD^2}{8v^2} \cdot \sec^2\theta = \frac{D}{2}\tan\theta + \frac{\sqrt{3}D}{2}$		
$\frac{gD^2}{8v^2} \cdot \sec^2\theta = \frac{1}{2}D(\tan\theta + \sqrt{3})$		
$\frac{gD}{8v^2} \cdot \sec^2\theta = \frac{1}{2}(\tan\theta + \sqrt{3})$		
$\therefore D = \frac{\frac{1}{2}8v^2(\tan\theta + \sqrt{3})}{g \sec^2\theta}$		① D or x in terms of $\tan\theta$.
$D = \frac{4v^2}{g} \cdot \cos^2\theta(\tan\theta + \sqrt{3})$		
$D = \frac{4v^2}{g}(\cos^2\theta\tan\theta + \sqrt{3}\cos^2\theta)$		
$D = \frac{4v^2}{g}(\sin\theta\cos\theta + \sqrt{3}\cos^2\theta$		
$D = \frac{4v^2}{g}\cos\theta(\sin\theta + \sqrt{3}\cos\theta)$.		① correct solution
(iii) $D = \frac{4v^2}{g}\left[\cos\theta(\sin\theta + \sqrt{3}\cos\theta)\right]$		
$= \frac{2v^2}{g}(2\sin\theta\cos\theta + 2\sqrt{3}\cos^2\theta)$	2	① Derivative
$= \frac{2v^2}{g}(\sin 2\theta + \sqrt{3}(2\cos^2\theta))$		① change to 2θ terms.
$= \frac{2v^2}{g}(\sin 2\theta + \sqrt{3}(1 + \cos 2\theta))$		
$\frac{dD}{d\theta} = \frac{2v^2}{g}(2\cos 2\theta - 2\sqrt{3}\sin 2\theta)$		
$= \frac{4v^2}{g}(\cos 2\theta - \sqrt{3}\sin 2\theta)$		